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Semiprime Modules with Maximum Conditions

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A module $_RM$ is semiprime if for each $0 \neq m \in M$ there exists $f \in \mathrm{Hom}_R(M, R)$ with $(mf)m \neq 0$. In Section 1 semiprime artinian modules are seen to be isomorphic to finite direct sums of minimal left ideals generated by idempotents. Semiprime noetherian modules have endomorphism rings which are left orders in semisimple artinian rings; and necessary and sufficient conditions for the latter situation to occur are given in Section 3. Prime modules are defined analogously and are treated simultaneously; and the above results are actually considered in the broader milieu of Morita contexts. In Sections 4 and 5 the classical density theorem for rings with faithful minimal left ideals is generalized (with a weakened definition of density) to include semiprime rings possessing faithful finite dimensional left ideals. The method of proof covers the infinite dimensional case as well. As a consequence, the classical density theorem is extended to rings with faithful completely reducible left ideals. In Section 6, the endomorphism ring of a torsionless module over a dense ring of transformations is shown to be a ring of the same type.

PRELIMINARIES

We do not assume that rings contain identity elements, except as indicated. An effort will be made to consistently write homomorphisms on the side opposite to the scalars. Given a ring R, R^1 will denote $R \oplus Z$ with the customary multiplication. By a *Morita context* $\mathcal{M} = (R, M, N, S)$ one means rings R, S and bimodules $_RM_S$ and $_SN_R$, together with bimodule homomorphisms $(\ ,\ ) : M \otimes_S N \to R$ and $[\ ,\ ] : N \otimes_R M \to S$ which satisfy $(m, n)\, m' = m[n, m']$, $[n, m]\, n' = n(m, n')$ for all $m, m' \in M$, $n, n' \in N$. By the *standard Morita context* for $_RM$ we mean the context (R, M, N, S) with $N = \mathrm{Hom}_R(M, R)$, $S = \mathrm{Hom}_R(M, M)$, where $(m, f) = mf$ and $[f, m]$ is defined by $m'[f, m] = (m', f)m$ for $f \in N$, $m, m' \in M$. For A any subset of M and B any subset of N we set $(A, B) = \{\sum_{i=i}^{t} (a_i, b_i) \mid a_i \in A, b_i \in B\}$ and $[B, A] = \{\sum_{i=1}^{t} [b_i, a_i] \mid a_i \in A, b_i \in B\}$. We will be concerned with various nonsingularity conditions on a context $\mathcal{M} = (R, M, N, S)$. For example, when (,) is nonsingular in the first variable (i.e., $(m, N) \neq 0$ for $0 \neq m \in M$) then ${}_RM$ is a torsionless module.

A module ${}_RM$ is *finite dimensional* if it contains no infinite direct sums of nonzero submodules. A nonzero module ${}_RM$ is *uniform* if any two nonzero submodules have nonzero intersection. Evidently, a nonzero finite dimensional module contains uniform submodules [3, Theorem 1.1]. For all notions concerning essential submodules, singular submodules, and other undefined terms, we refer the reader to [9].

The author wishes to acknowledge his debt to the treatment of Morita contexts in [1]. This paper is a lineal descendent of that effort. The reader familiar with this subject is advised to begin his reading with Theorem 1.2 and Proposition 3.3, referring back to earlier material as required.

1. The Morita context (R, M, N, S) is said to be *prime* (*semiprime*) if for every $0 \neq m \in M$, $0 \neq m_1 \in M$, $(m, N)\, m_1 \neq 0$ ($(m, N)m \neq 0$). A module ${}_RM$ is called *prime* (*semiprime*) when the standard Morita context for ${}_RM$ is prime (semiprime). Observe that if there exists a semiprime (prime) Morita context (R, M, N, S) then ${}_RM$ is semiprime (prime).

We remark that prime modules in this sense are prime in the sense of [7] but not conversely, as we are restricting our attention to torsionless modules. Evidently, submodules and direct products of semiprime modules are again semiprime. It follows that torsionless modules over semiprime rings are semiprime modules. The next proposition shows that these are essentially all the semiprime modules.

PROPOSITION 1.1. *For a Morita context $\mathcal{M} = (R, M, N, S)$ the following conditions are equivalent.*

(1) *$\mathcal{M}$ is prime (semiprime).*

(2) *$\langle R\rangle = R/\mathrm{ann}\, {}_RM$ is a prime (semiprime) ring and $m[N, M] \neq 0$ for all $0 \neq m \in M$.*

(3) *$\langle S\rangle = S/\mathrm{ann}\, M_S$ is a prime (semiprime) ring and $(M, N)m \neq 0$ for all $0 \neq m \in M$.*

Proof. Here $\mathrm{ann}\, {}_RM$ denotes the annihilator of M in R; i.e., $\mathrm{ann}\, {}_RM = \{r \in R \mid rM = 0\}$. Assume that $\mathcal{M}$ is prime and let $\langle 0\rangle \neq \langle a\rangle \in \langle R\rangle$, $\langle 0\rangle \neq \langle b\rangle \in \langle R\rangle$ be given, where $\langle a\rangle = a + \mathrm{ann}\, {}_RM$ for $a \in R$. Then $aM \neq 0$, $bM \neq 0$. Sunce $\mathcal{M}$ is prime $0 \neq (aM, N)\, bM = a(M, N)\, bM$. In particular, $\langle 0\rangle \neq \langle a(M, N)b\rangle \subseteq \langle a\rangle\langle R\rangle\langle b\rangle$, and this proves that $\langle R\rangle$ is prime. And clearly $m[N, M] \neq 0$ for any $0 \neq m \in M$. Thus (1) implies (2) for $\mathcal{M}$ prime.

Now suppose that (2) holds with $\langle R\rangle$ prime, and let $0 \neq m \in M$, $0 \neq m_1 \in M$

be given. Then by hypothesis $(m, N)M \neq 0$ and $(m_1, N)M \neq 0$ so that $\langle(m, N)\rangle \neq \langle 0\rangle$ and $\langle(m_1, N)\rangle \neq \langle 0\rangle$. Since $\langle R\rangle$ is a prime ring $\langle 0\rangle \neq \langle(m, N)\rangle\langle R\rangle\langle(m_1, N)\rangle \subseteq \langle(m, NR)m_1, N)\rangle$. In particular $(m, N)m_1 \neq 0$, proving that $\mathcal{M}$ is prime. Thus (2) implies (1) for $\langle R\rangle$ prime. The semiprime case is treated by taking $a = b$, $m = m_1$ in the previous paragraphs.

To check that (1) and (3) are equivalent, one need only consider the opposite Morita context $\mathcal{N} = (S^\circ, M^\circ, N^\circ, R^\circ)$ and observe that $\mathcal{M}$ is prime (semiprime) if and only if $\mathcal{N}$ is. ▮

Our first result for semiprime modules can be regarded as a module-theoretic version of a portion of the usual Wedderburn theory for rings.

THEOREM 1.2. *For a module $_RM$, the following conditions are equivalent.*

(1) *$_RM$ is semiprime (prime) artinian.*

(2) *$_RM$ is a finite direct sum of (isomorphic) simple projective submodules.*

Proof. (1) *implies* (2). We first show that simple submodules of $_RM$ are projective direct summands of M. Let K be any simple submodule of M and $0 \neq m \in K$. Since $_RM$ is semiprime we can choose $f \in \mathrm{Hom}_R(M, R)$ with $(mf)m \neq 0$. f must be monic on K, so $(mf)(mf) \neq 0$. Hence $(Kf)^2 \neq 0$. By a well-known result [9; p. 62], $Kf = Re$ for some $e = e^2 \in R$, and so $K \simeq Kf$ is projective. Now $e = m_1 f$ for some $m_1 \in K$, and $(m_1 - (m_1 f)m_1)f = e - e^2 = 0$. Since f is monic on K, $m_1 = (m_1 f)\, m_1 = m_1[f, m_1]$. Thus $[f, m_1] \in \mathrm{Hom}_R(M, K)$ splits the inclusion map $0 \to K = Rm_1 \to M$, proving that K is a direct summand of M.

From this it follows directly that $M = K_1 \oplus \cdots \oplus K_t$ where each $_RK_i$ is simple and projective. If additionally $_RM$ is prime, then given $0 \neq m_i \in K_i$ and $0 \neq m_j \in K_j$ for any choice of i and j there exists $f \in \mathrm{Hom}_R(M, R)$ with $(m_i f)\, m_j \neq 0$. Hence $0 \neq [f, m_j]\,|_{K_i} \in \mathrm{Hom}_R(K_i, K_j)$. From this it is clear that K_i is isomorphic to K_j.

(2) *implies* (1). Suppose that $_RM = K_1 \oplus \cdots \oplus K_t$ where each K_j is a projective simple module. Then each K_j is isomorphic to a minimal left ideal I_j of R generated by an idempotent. Each I_j is not nilpotent and hence is a semiprime R-module. It follows that $M \simeq I_1 \oplus \cdots \oplus I_t$ is a semiprime module. If all the I_j are isomorphic then $I_j I_k \neq 0$ for any choice of j and k, and hence $M \simeq I_1 \oplus \cdots \oplus I_t$ is prime. ▮

2. For a module $_RM$, let $\hat{M}$ denote the injective hull of $_RM$, $E(M) = \mathrm{Hom}_R(M, M)$, $P(M) = \{\alpha \in E(M) \mid \ker \alpha \text{ is an essential submodule of } {}_RM\}$. $P(M)$ is an ideal of $E(M)$. More generally, given a Morita context (R, M, N, S), set $P_S(M) = \{s \in S \mid l_M(s) \text{ is an essential submodule of } {}_RM\}$, where $l_M(s) = \{m \in M \mid ms = 0\}$. $P_S(M)$ is an ideal of S containing ann M_S. This is a good

place to note that our notation for annihilators is to use $l(\ )$ for left annihilators, for example $l_M(s)$ above, and $r(\ )$ for right annihilators; subscripts being omitted when no confusion can arise.

Given a Morita context (R, M, N, S) we may regard $\langle S \rangle = S/\text{ann}\, M_S$ as a subring of $E(M)$. Extension of elements of $\langle S \rangle$ to elements of $E(\hat{M})$ induces a ring monomorphism

$$S/P_S(M) \to E(\hat{M})/P(\hat{M}).$$

In the sequel, we will regard $S/P_S(M)$ as a subring of $E(\hat{M})/P(\hat{M})$. Of particular importance is the standard Morita context where $S = E(M)$.

Remark. When ${}_RM$ is finite dimensional (i.e., contains no infinite direct sums of nonzero submodules) then $E(\hat{M})$ is known to be semiperfect with radical $P(\hat{M})$ and with the dimension of $E(\hat{M})/P(\hat{M})$ equal to the dimension of ${}_RM$ [9, p. 103].

In particular in $E(M)/P(M)$, and more generally in $S/P_S(M)$, no chain of left (or right) annihilator ideals can have more than $d({}_RM) + 1$ terms where $d({}_RM)$ equals the dimension of ${}_RM$. This generalizes an observation made in [2, Theorem 1.3] and [11, Corollary 4.3] for rings.

It is known that $E(\hat{M})/P(\hat{M})$ is a regular self-injective ring [10]. We will need this information only in a special case, and it seems appropriate to provide a self-contained proof here. $Z({}_RM) = \{m \in M \mid l(m)$ is an essential left ideal of $R\}$ is the familiar *singular submodule* of ${}_RM$.

PROPOSITION 2.1. *Suppose that $Z({}_RM) = 0$. Then $P(M) = 0 = P(\hat{M})$ and $E(\hat{M})$ is a regular, left self-injective ring.*

Proof. The first observation is routine. By an easy calculation one observes that $E(\hat{M})$ is regular if and only if $\ker \varphi$ and $\hat{M}\varphi$ are direct summands of ${}_R\hat{M}$ for all $\varphi \in E(\hat{M})$. Let $\varphi \in E(\hat{M})$ be given. If ${}_RK$ is an essential extension of $\ker \varphi$ in $\hat{M}$, then for every $x \in K$, $(\ker \varphi : x)\, x\varphi = 0$, where $(\ker \varphi : x) = \{r \in R \mid rx \in \ker \varphi\}$. Since $Z({}_RM) = 0$ and $(\ker \varphi : x)$ is an essential left ideal of R, $x\varphi = 0$. Thus $K = \ker \varphi$, proving that $\ker \varphi$ is closed in ${}_R\hat{M}$. It follows that $\ker \varphi$ is injective, and consequently so is $\hat{M}\varphi \cong \hat{M}/\ker \varphi$.

Next, to show that $E = E(\hat{M})$ is left self-injective, let $\theta \in \text{Hom}_E(J, E)$ be given where J is a left ideal of E. Note that $\ker \mu \subseteq \ker \mu^\theta$ for any $\mu \in J$. For we know that $\ker \mu$ is a direct summand of $\hat{M}$. Letting π denote the natural projection of $\hat{M}$ onto $\ker \mu$, $\pi\mu = 0$ so that $\pi\mu^\theta = (\pi\mu)^\theta = 0$ and $(\ker \mu)\, \mu^\theta = \hat{M}\pi\mu^\theta = 0$.

Define $\theta' \in \text{Hom}_R(MJ, M)$ via $(\sum_{i=1}^t m_i\mu_i)\, \theta' = \sum_{i=1}^t m_i\mu_i^\theta$, for any $m_i \in M$, $\mu_i \in J$. We claim that θ' is well defined. If $\sum_{i=1}^t m_i\mu_i = \sum_{j=1}^s n_j\nu_j$ with the $n_j \in M$, $\nu_j \in J$, choose $\omega \in J$ so that $\sum_{i=1}^t E\mu_i + \sum_{j=1}^s E\nu_j = E\omega$. This is possible since E is a regular ring. Write $\mu_i = \mu_i'\omega$, $\nu_j = \nu_j'\omega$ for some

$\mu_i' \in E$, $\nu_j' \in E$. Then $\mu_i^\theta = \mu_i'\omega^\theta$, $\nu_j^\theta = \nu_j'\omega^\theta$. Since $\ker \omega \subseteq \ker \omega^\theta$ and $(\sum_{i=1}^t m_i\mu_i' - \sum_{j=1}^s n_j\nu_j')\omega = 0$, we have $(\sum_{i=1}^t m_i\mu_i' - \sum_{j=1}^s n_j\nu_j')\,\omega^\theta = 0$; i.e., $\sum_{i=1}^t m_i\mu_i^\theta = \sum_{j=1}^s n_j\nu_j^\theta$. θ' is clearly an R-homomorphism, and so extends to a homomorphism $\alpha \in E$ (this is the only place where the injectivity of ${}_R\hat{M}$ is used). It is easy to check that $\mu^\theta = \mu \circ \alpha$ for all $\mu \in J$. Hence θ extends to an element of $\mathrm{Hom}_E(E, E)$, proving that ${}_EE$ is injective. ∎

COROLLARY 2.2. *If ${}_RM$ is finite dimensional and $Z({}_RM) = 0$ then $E(\hat{M})$ is a semisimple artinian ring.*

Proof. Obviously $d({}_RM) = d({}_R\hat{M})$, and (using the fact that $E(\hat{M})$ is regular) it is easy to see that $d({}_{\hat{E}}\hat{E}) \leqslant d({}_R\hat{M})$. ${}_{\hat{E}}\hat{E}$ is therefore finite dimensional and a regular ring, and so is semisimple artinian. ∎

We will eventually be concerned with conditions on a semiprime module ${}_RM$ which are sufficient to guarantee that $Z({}_RM) = 0$. But first we need some information about the ideal $P(M)$.

LEMMA 2.3. *Let (R, M, N, S) be a Morita context. Then $[N, Z({}_RM)] \subseteq P_S(M)$ and $MP_S(M) \subseteq Z({}_RM)$.*

Proof. We consider a generator $[n, z]$ of $[N, Z({}_RM)]$, $n \in N$, $z \in Z({}_RM)$. Given $0 \neq m \in M$, either $(m, n) = 0$ or there exists $a \in R^1$ with $0 \neq a(m, n) \in l(z)$. Hence either $m[n, z] = 0$ or $am[n, z] = 0$, and this proves that $l_M([n, z])$ is an essential submodule of ${}_RM$. The second inclusion is safely left to the reader. ∎

PROPOSITION 2.4. *Let (R, M, N, S) be a Morita context such that $(M, N)m \neq 0$ for every $0 \neq m \in M$; $\langle S\rangle = S/\mathrm{ann}\, M_S$.*

(1) *For K any submodule of ${}_RM$, $d({}_{\langle S\rangle}\langle[N, K]\rangle) = d({}_RK)$.*

(2) *For A a left ideal of S, $d({}_RMA) = d({}_{\langle S\rangle}\langle A\rangle)$. In particular, when $1 \in S$ and M_S is unitary, $d({}_RM) = d({}_{\langle S\rangle}\langle S\rangle)$.*

(3) *If K is an essential submodule of ${}_RM$, then $\langle[N, K]\rangle$ is an essential left ideal of $\langle S\rangle$.*

(4) *If A is a left ideal of S such that $\langle A\rangle$ is an essential left ideal of $\langle S\rangle$, then MA is an essential submodule of ${}_RM$.*

(5) $Z({}_{\langle S\rangle}\langle S\rangle) = \langle P_S(M)\rangle$.

(6) $d({}_RZ({}_RM)) = d({}_{\langle S\rangle}\langle P_S(M)\rangle)$. *In particular, $Z({}_RM) = 0$ if and only if $P_S(M) = \mathrm{ann}\, M_S$.*

Proof. (1) Let $\{A_i \mid i \in I\}$ be a family of left ideals of S with each $\langle A_i\rangle \neq \langle 0\rangle$, $\langle A_i\rangle \subseteq \langle[N, K]\rangle$, and $\sum_{i\in I} \langle A_i\rangle$ a direct sum. Then for each

$i \in I$, $0 \neq MA_i \subseteq K$. The sum $\sum_{i \in I} MA_i$ is seen to be direct since $\langle 0 \rangle \neq \langle [N, m] \rangle \subseteq \langle A_i \rangle$ for any $0 \neq m \in MA_i$. Thus $d({}_{\langle S \rangle}\langle [N, K] \rangle) \leqslant d({}_R K)$. For the reverse inequality, one proceeds along a similar route showing that a direct sum $\sum_{j \in J} K_j$ of nonzero submodules K_j of ${}_R M$ induces a direct sum $\sum_{j \in J} \langle [N, K_j] \rangle$ of nonzero left ideals $\langle [N, K_j] \rangle$ of $\langle S \rangle$ contained in $\langle [N, K] \rangle$.

(2) can be proved directly, using the proof of (1) as a model. Alternatively, we note that from the given hypothesis, it is evident that $r_{\langle S \rangle}(\langle [N, M] \rangle) = 0$. For if $\langle 0 \rangle \neq \langle \alpha \rangle \in \langle S \rangle$, $M\alpha \neq 0$, and hence $(M, N)\, M\alpha \neq 0$. In particular, $0 \neq \langle [N, M] \rangle \langle \alpha \rangle$. Then from (1) we have $d_R(MA) = d({}_{\langle S \rangle}\langle [N, MA] \rangle) = d({}_{\langle S \rangle}\langle [N, M] \rangle \langle A \rangle) = d({}_{\langle S \rangle}\langle A \rangle)$, the latter equality holding since $\langle [N, M] \rangle \langle A \rangle$ is an essential submodule of ${}_{\langle S \rangle}\langle A \rangle$.

(3) Let K be an essential submodule of ${}_R M$ and let $\langle 0 \rangle \neq \langle \alpha \rangle \in \langle S \rangle$. Then $M\alpha \neq 0$, so $M\alpha \cap K \neq 0$. Therefore $0 \neq M[N, M\alpha \cap K]$, i.e., $\langle 0 \rangle \neq \langle [N, M\alpha \cap K] \rangle \subseteq \langle [N, M] \rangle \langle \alpha \rangle \cap \langle [N, K] \rangle$, proving that $\langle [N, K] \rangle$ is an essential left ideal of $\langle S \rangle$.

(4) Let $0 \neq m \in M$ be given. Then $\langle [N, m] \rangle \neq \langle 0 \rangle$ so that $\langle [N, m] \rangle \cap \langle A \rangle \neq \langle 0 \rangle$. Choose $n \in N$ and $\alpha \in A$ with $\langle 0 \rangle \neq \langle [n, m] \rangle = \langle \alpha \rangle$. Then $0 \neq (M, n)m = M[n, m] = M\alpha \subseteq MA$.

(5) Let $\langle \alpha \rangle \in Z({}_{\langle S \rangle}\langle S \rangle)$. Then $l_{\langle S \rangle}(\langle \alpha \rangle)$ is an essential left ideal of $\langle S \rangle$. Hence by (4), MA is an essential submodule of ${}_R M$, where

$$A = \{\beta \in S \mid \beta\alpha \in \operatorname{ann} M_S\}.$$

Also $MA\alpha = 0$, so that $\alpha \in P_S(M)$, and proving that $Z({}_{\langle S \rangle}\langle S \rangle) \subseteq \langle P_S(M) \rangle$.

For the reverse inclusion let $\alpha \in P_S(M)$, and set $K = l_M(\alpha)$. By (3) $\langle [N, K] \rangle$ is an essential left ideal of $\langle S \rangle$, and $\langle [N, K] \rangle \langle \alpha \rangle = \langle [N, K\alpha] \rangle = \langle 0 \rangle$, so that $\alpha \in Z({}_{\langle S \rangle}\langle S \rangle)$.

(6) This follows directly from the proofs of (2) and (3), and Lemma 2.3. ∎

Remark. One can improve (3) and (4) for Morita contexts as in the previous theorem to the following.

(3′) If K and L are submodules of ${}_R M$ with K an essential R-submodule of L then $\langle [N, K] \rangle$ is an essential $\langle S \rangle$-submodule of $\langle [N, L] \rangle$.

(4′) If A and B are left ideals of S with $\langle A \rangle$ an essential $\langle S \rangle$-submodule of $\langle B \rangle$ then MA is an essential R-submodule of MB.

3. We are now ready to give some sufficient conditions for the singular submodule to be zero.

PROPOSITION 3.1. *Suppose that (R, M, N, S) is a Morita context with ${}_R M$ noetherian. Then $\langle P_S(M) \rangle$ is a nilpotent ideal of $\langle S \rangle$. If, additionally, ${}_R M$ is semiprime, then $Z({}_R M) = 0$.*

Proof. Set $P = P_S(M)$ and suppose that $\langle P\rangle^{n+1} \neq 0$, $n \geqslant 0$. (Here $\langle P\rangle^{\circ} = \langle S\rangle^1$.) Then choose $\langle 0\rangle \neq \langle\alpha\rangle \in \langle P\rangle$ so that $\langle\alpha\rangle\langle P\rangle^n \neq 0$ and with $l_M(\alpha)$ maximal among such α. For any $\beta \in P$, $l_M(\beta) \cap M\alpha \neq 0$ since $l_M(\beta)$ is an essential submodule of ${}_RM$. It follows that $l_M(\alpha) \subsetneqq l_M(\alpha\beta)$. By the choice of α it must be the case that $\langle\alpha\rangle\langle\beta\rangle\langle P\rangle^n = 0$. Since β was arbitrary in P, $\langle\alpha\rangle\langle P\rangle^{n+1} = 0$. Thus, were $\langle P\rangle$ not nilpotent we would get an infinite chain

$$l_{\langle S\rangle}(\langle P\rangle) \subsetneqq l_{\langle S\rangle}(\langle P\rangle^2) \subsetneqq \cdots.$$

This clearly induces a corresponding chain

$$l_M(P) \subsetneqq l_M(P^2) \subsetneqq \cdots,$$

which violates the hypothesis that ${}_RM$ is noetherian. Therefore $\langle P\rangle$ must be nilpotent.

If ${}_RM$ is semiprime, then by Proposition 1.1, $\langle S\rangle$ is a semiprime ring and so $\langle P\rangle = \langle 0\rangle$. Thus $P = \operatorname{ann} {}_RM$, and then $Z({}_RM) = 0$ by the previous proposition. ∎

Remark. In the situation of the above proposition, nil subrings of $\langle S\rangle$ are nilpotent (with bounded nilpotency index). For by the remark preceding Proposition 2.1, nil subrings of $S/P_S(M) \cong \langle S\rangle/\langle P_S(M)\rangle$ are nilpotent of index $\leqslant d({}_RM) + 1$, and $\langle P_S(M)\rangle$ is nilpotent when ${}_RM$ is noetherian. This indicates an alternate proof that nil subrings of the endomorphism ring of a noetherian module are nilpotent of bounded index, a result also provided by Lance Small and Joe Fisher.

PROPOSITION 3.2. *If ${}_RM$ is semiprime and R satisfies the maximum condition on annihilators of elements of M then $Z({}_RM) = 0$.*

Proof. If $Z({}_RM) \neq 0$ choose $0 \neq m \in Z({}_RM)$ with $l(m)$ maximal among all such m. Since M is semiprime there exists $f \in \operatorname{Hom}_R(M, R)$ with $0 \neq (mf)m$. On the other hand, $l(m) \cap R(m, f) \neq 0$, so there exists $a \in R$ with $0 \neq a(m, f) \in l(m)$. But then $l(m) \subsetneqq l((m, f)m)$ violating the choice of m. ∎

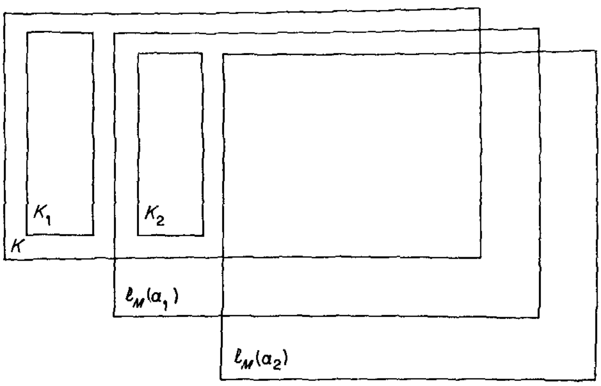
Observe that there is a close analogy between the two previous propositions. Indeed one can provide proofs along either of the two lines indicated above. Thus, for example, one can conclude that if ${}_RM$ is semiprime and satisfies the maximum condition on $\{l_M(\alpha) \mid \alpha \in P(M)\}$ (these are kernels of homomorphisms) then $Z({}_RM) = 0$.

The hypothesis that R satisfy the maximum condition on annihilators of subsets of M is not a common one. It is therefore worth pointing out that for a Morita context (R, M, N, S) satisfying $(m, N) \neq 0$ for $0 \neq m \in M$, this hypothesis is satisfied when R satisfies the maximum condition on left annihilator ideals.

PROPOSITION 3.3. *Let (R, M, N, S) be a semiprime Morita context, and suppose that K is a finite dimensional submodule of ${}_RM$ with $Z({}_RK) = 0$. Then there exists $\alpha \in [N, K]$ with $l_M(\alpha) \cap K = 0$.*

Proof. Choose $0 \neq m_1 \in K_1'$, for K_1' any uniform submodule of ${}_RK$. There exists $n_1 \in N$ with $(m_1, n_1) m_1 \neq 0$. Set $\alpha_1 = [n_1, m_1] \in [N, K]$. Now $0 \neq M\alpha_1 \subseteq K_1'$, and $0 \neq K_1'\alpha_1$ since $m_1 \in K_1'$. Since $Z({}_RK_1') = 0$, it must be the case that $l_M(\alpha_1) \cap K_1' = 0$. Choose a submodule ${}_RK_1$ with $K_1' \subseteq K_1 \subseteq K$ maximal with respect to $K_1 \cap l_M(\alpha_1) = 0$. Then $K_1 + (l_M(\alpha_1) \cap K)$ (direct sum) is an essential submodule of K.

If $l_M(\alpha_1) \cap K = 0$, we go no further. If $l_M(\alpha_1) \cap K \neq 0$, we choose $0 \neq m_2 \in K_2'$, where K_2' is a uniform submodule of $l_M(\alpha_1) \cap K$. Choose $n_2 \in N$ with $(m_2, n_2) m_2 \neq 0$. As above, we set $\alpha_2 = [n_2, m_2]$ and note that $l_M(\alpha_2) \cap K_2' = 0$. So it is possible to choose a submodule $K_2' \subseteq K_2 \subseteq l_M(\alpha_1) \cap K$ maximal with respect to $K_2 \cap l_M(\alpha_2) = 0$. Note that $K_1 + K_2 + (l_M(\alpha_1) \cap l_M(\alpha_2) \cap K)$ (direct sum) is an essential submodule of K, and $K_2\alpha_1 = 0$. The diagram below illustrates the construction thus far.



We may continue this construction. After t repetitions we have submodules $K_i \subseteq K$ $(i = 1, 2, \ldots, t)$ and elements $\alpha_i \in [N, K_i]$ such that the direct sum $K_1 + \cdots + K_t + (\bigcap_{i=1}^t l_M(\alpha_i) \cap K)$ is an essential submodule of K, $M\alpha_i \subseteq K_i$, $l_M(\alpha_i) \cap K_i = 0$, and $K_i\alpha_j = 0$ for $i > j$. Since ${}_RM$ is finite dimensional, we may assume that $\bigcap_{i=1}^t l_M(\alpha_i) \cap K = 0$.

Set $\alpha = \alpha_1 + \cdots + \alpha_t \in [N, K]$. It remains to show that $l_M(\alpha) \cap K = 0$, and for this it is enough to show that $l_M(\alpha) \cap (K_1 + \cdots + K_t) = 0$. Let $k_i \in K_i$ $(i = 1, \ldots, t)$ and suppose that

$$\begin{aligned} 0 = (k_1 + \cdots + k_t)\alpha &= (k_1 + \cdots + k_t)(\alpha_1 + \cdots + \alpha_t) \\ &= k_1\alpha_1 + (k_1 + k_2)\alpha_2 + \cdots + (k_1 + \cdots + k_t)\alpha_t . \end{aligned}$$

Since $M\alpha_i \subseteq M[N, K_i] \subseteq K_i$ and the sum $K_1 + \cdots + K_t$ is direct, we have

$$\begin{aligned} k_1\alpha_1 &= 0 \\ (k_1 + k_2)\,\alpha_2 &= 0 \\ &\vdots \\ (k_1 + \cdots + k_t)\,\alpha_t &= 0. \end{aligned}$$

Solving these equations successively using the fact that $l_M(\alpha_i) \cap K_i = 0$, we find that $k_1 = k_2 = \cdots = k_t = 0$. Thus $l_M(\alpha) \cap K = 0$. ∎

Remarks. (1) For future reference we record that α in Proposition 3.3 was chosen as follows. $\alpha = \sum_{i=1}^{t} [n_i, m_i]$, $n_i \in N$, $m_i \in K$, with $\sum_{i=1}^{t} Rm_i$ and $K\alpha$ essential submodules of ${}_RK$.

(2) In the previous proposition one could replace the semiprime hypothesis with the assumption that given any submodule L of ${}_RK$ there exists $\alpha \in S$ with $M\alpha \subseteq L$ and $L\alpha \neq 0$, and arrive at the same conclusion. For it was only to obtain this property that semiprimeness was used.

Let us now assume that the Morita context in Proposition 3.3 satisfies the following condition in place of semiprimeness: given any $m \in K$ there exists $n \in N$ with $m[n, m] = m$. Then each $\alpha_i = [n_i, m_i]$ in the proof can be chosen so that $\alpha_i^2 = \alpha_i \in [N, K]$. Also $\alpha_i\alpha_j = [n_i, m_i]\,\alpha_j = [n_i, m_i\alpha_j] = 0$ for $i > j$. At this point a standard idempotent calculation [9, p. 68] shows that there exists $\beta = \beta^2 \in [N, K]$ with $\sum_{i=1}^{t} M\alpha_i = M\beta$. It follows that $M\beta = K$ and that $\beta\,|_K = 1_K$. We summarize this below for later use.

COROLLARY 3.4. *Let (R, M, N, S) be a Morita context, and K a finite dimensional submodule of ${}_RM$ with $Z({}_RK) = 0$. Suppose that given any $m \in K$ there exists $n \in N$ with $(m, n)m = m$. Then there exists $\beta = \beta^2 \in [N, K]$ with $\beta\,|_K = 1_K$.*

THEOREM 3.5. *Let $\mathscr{M} = (R, M, N, S)$ be a Morita context satisfying $(M, N)m \neq 0$ for all $0 \neq m \in M$. Then $S/\operatorname{ann} M_S$ has a semisimple [simple] artinian classical left quotient ring (necessarily isomorphic to $E(\hat{M})$) if and only if $\mathscr{M}$ is semiprime [prime], ${}_RM$ is finite dimensional, and $Z({}_RM) = 0$.*

Proof. Suppose that $\langle S\rangle = S/\operatorname{ann} M_S$ has a semisimple [simple] artinian classical left quotient ring. Then, as in [3], one learns that $\langle S\rangle$ is a semiprime [prime] ring, ${}_{\langle S\rangle}\langle S\rangle$ is finite dimensional, and $Z({}_{\langle S\rangle}\langle S\rangle) = 0$. By Proposition 1.1, $\mathscr{M}$ is semiprime [prime]. By Proposition 2.4, ${}_RM$ is finite dimensional and $Z({}_RM) = 0$.

Conversely, suppose that $\mathscr{M}$ is semiprime [prime], ${}_RM$ is finite dimensional, and $Z({}_RM) = 0$. By Proposition 1.1, $\langle S\rangle$ is semiprime [prime]. It is evident that $P(\hat{M}) = 0$, and $P_S(M) = \operatorname{ann} M_S$ by Proposition 2.4. Hence we can regard $\langle S \rangle = S/\text{ann}\, M_S$ as a subring of $E(\hat{M})$, a semisimple artinian ring (Corollary 2.2). Let $0 \neq \varphi \in E(\hat{M})$ be given, and set $K = M\varphi^{-1} \cap M$. K is an essential submodule of ${}_RM$. By Proposition 3.3 there exists $\alpha \in [N, K]$ with $l_M(\alpha) \cap K = 0$. Then clearly $l_M(\alpha) = 0$ and $(M\alpha)\varphi \subseteq K\varphi \subseteq M$. To complete the proof it suffices to show that $\langle \alpha \rangle \varphi \in \langle S \rangle$, that $\langle \alpha \rangle$ is invertible in $E(\hat{M})$, and that regular elements of $\langle S \rangle$ are invertible in $E(\hat{M})$. [Additionally one these properties are proved, it is easy to check that $E(\hat{M})$ must be a simple ring when $\langle S \rangle$ is prime.]

For any $n \in N$ and $m \in K$, $\langle [n, m] \rangle \varphi = \langle [n, m\varphi] \rangle$ since they agree on M. Hence $\langle [N, K] \rangle \varphi = \langle [N, K\varphi] \rangle \subseteq \langle S \rangle$. In particular $\langle \alpha \rangle \varphi \in \langle S \rangle$. Next $\langle \alpha \rangle$ is monic on ${}_R\hat{M}$, so ${}_R\hat{M}\langle \alpha \rangle$ is injective. On the other hand $M\langle \alpha \rangle$ is an essential submodule of ${}_RM$ (see Remark (1) following Proposition 3.3). It follows that $\hat{M}\langle \alpha \rangle = \hat{M}$. Hence $\langle \alpha \rangle$ is invertible in $E(\hat{M})$, and this argument in fact shows that any monic element of $\langle S \rangle$ is invertible in $E(\hat{M})$. Finally, any regular element $\langle \beta \rangle \in \langle S \rangle$ must be monic (otherwise use Proposition 3.3 to choose $0 \neq \langle \gamma \rangle \in \langle S \rangle$ with $M\gamma \subseteq l_M(\beta)$) and hence invertible in $E(\hat{M})$. ▮

Propositions 3.1 and 3.2 yield the following consequences.

COROLLARY 3.6. *If (R, M, N, S) is a semiprime [prime] Morita context with ${}_RM$ Noetherian, then $S/\text{ann}\, M_S$ has a semisimple [simple] classical left quotient ring isomorphic to $E(\hat{M})$.*

COROLLARY 3.7. *If (R, M, N, S) is a semiprime [prime] Morita context with ${}_RM$ finite dimensional and with R satisfying the maximum condition on annihilators of elements of M, then $S/\text{ann}\, M_S$ has a semisimple [simple] classical left quotient ring isomorphic to $E(\hat{M})$.*

Remarks. (1) The reader concerned primarily with Theorem 3.5 should observe that it rests only on a fraction of the prior development. Specifically, Proposition 1.1, Corollary 2.2, part of Proposition 2.4, and Proposition 3.3.

(2) For R a ring with identity and Morita contexts (R, M, N, S) satisfying $(M, N) = R$ (i.e., ${}_RM$ is a generator), the Morita contexts of Corollary 3.7 coincide with those of Theorem 3.5. For if $Z({}_RM) = 0$ and ${}_RM$ is a finite dimensional generator, then $Z({}_RR) = 0$ and ${}_RR$ is finite dimensional. It then follows that left annihilator ideals of R are closed [13, Lemma 2], and consequently any chain of left annihilator ideals of R must be finite.

(3) One might reasonably ask for more information as to the nature of the module in Theorem 3.5. When $\hat{R}$ is an artinian left quotient ring of R (here we mean quotient ring in the sense of [9]), then a finite dimensional torsionless module ${}_RM$ with $Z({}_RM) = 0$ is isomorphic to a submodule of a finitely generated free module. (See Proposition 17 and Theorem 18 of [14].)

(4) Some special cases of Theorem 3.5 appear in [4] and [12].

COROLLARY 3.8. [3, Theorem 13]. *A ring R has a classical left quotient ring which is semisimple [simple] artinian if and only if R is semiprime [prime], $d({}_RR) < \infty$ and R satisfies the maximum condition on left annihilators of elements.*

Proof. Apply Theorem 3.5 and Corollary 3.7 to the context (R, R, R, R) with all maps given by multiplication in R. ▮

4. Let Δ_i, $i \in I$, be a division ring and V_i a Δ_i-vector space. Set $V = \bigoplus_{i\in I} V_i$, and let π_i denote the projection map of V onto V_i. If U is a submodule of ${}_\Delta V$, we define $\dim_\Delta U = \sum_{i\in I} \dim_{\Delta_i} U\pi_i$. Observe that $U\pi_i = \Delta_i U = U \cap V_i$, so that $U = \bigoplus_{i\in I} (U \cap V_i)$. Since $\mathrm{Hom}_\Delta(V_i, V_j) = 0$ for $i \neq j$, we can and will identify $\mathrm{Hom}_\Delta(V, V)$ with $\prod_{i\in I} \mathrm{Hom}_{\Delta_i}(V_i, V_i)$.

Let Δ and V be as in the preceding paragraph. A subring R of $\mathrm{Hom}_\Delta(V, V)$ is said to be a *dense* subring of $\mathrm{Hom}_\Delta(V, V)$ if given any $\tau \in \mathrm{Hom}_\Delta(V, V)$ and any finite dimensional submodule ${}_\Delta U$ of V there exist $r, s \in R$ with $r\tau = s$, $Vr \subseteq U$, and $r|_U$ an automorphism of ${}_\Delta U$. If r above can always be chosen with $r|_U = 1_U$, then we will say that R is a *classically dense* subring of $\mathrm{Hom}_\Delta(V, V)$.

THEOREM 4.1. *The following conditions on a ring R are equivalent.*

(1) *R is semiprime [prime], $Z({}_RR) = 0$, and R has a faithful finite dimensional left ideal.*

(2) *R has a faithful finite dimensional semiprime [prime] module ${}_RM$ with $Z({}_RM) = 0$.*

(3) *There exist division rings Δ_i and Δ_i-vector spaces V_i for $i = 1,..., n$ [$n = 1$] such that R can be embedded as a dense subring of $\bigoplus_{i=1}^n \mathrm{Hom}_{\Delta_i}(V_i, V_i)$.*

In the proof of this and succeeding theorems the prime case will be treated simultaneously, with the relevant statements surrounded by square parentheses. First we require a lemma.

LEMMA 4.2. *Suppose that ${}_RM$ is a faithful semiprime left R-module with $Z({}_RM) = 0$. Then so is any essential submodule of ${}_RM$.*

Proof. Let K be any essential submodule of ${}_RM$ and let $0 \neq a \in R$ be given. Let (R, M, N, S) be the standard Morita context for ${}_RM$. Then $aM \neq 0$ so that $(aM, N)\,aM \neq 0$, and in particular $Na \neq 0$. Since $Z({}_RM) = 0$ and K is an essential submodule of ${}_RM$, $0 \neq (K, Na) \subseteq (K, N) \cap Ra$. Since R is semiprime (Proposition 1.1), $0 \neq ((K, N) \cap Ra)^2 \subseteq Ra(K, N)$. In particular, $aK \neq 0$. ▮

Proof of Theorem 4.1. (1) *implies* (2) is clear.

(2) *implies* (3). Since ${}_RM$ is finite dimensional we can choose uniform submodules $M_1, \ldots, M$ of ${}_RM$ so that ${}_RM$ is an essential extension of $\bigoplus_{i=1}^m M_i$ [3, Theorem 1.1]. Renumbering if necessary, we have

$$M = M_1 \oplus \cdots \oplus M_n \oplus M_{n+1} \oplus \cdots \oplus M_m$$

where $\mathrm{Hom}_R(M_i, M_j) = 0$ for $1 \leqslant i \neq j \leqslant n$, and for each $k > n$ there exists $i_k \leqslant n$ with $\mathrm{Hom}_R(M_{i_k}, M_k) \neq 0$. [When ${}_RM$ is prime, $M_i[N, M_j] \neq 0$ where $N = \mathrm{Hom}_R(M, R)$ implies that $\mathrm{Hom}_R(M_i, M_j) \neq 0$ for $i \neq j$, and so $n = 1$.] Observe that since ${}_RM_{i_k}$ is uniform and $Z({}_RM_k) = 0$, any nonzero element of $\mathrm{Hom}_R(M_{i_k}, M_k)$ must be a monomorphism. Set

$$M_0 = M_1 \oplus \cdots \oplus M_n;$$

$Z({}_RM_0) = 0$ and ${}_RM_0$ is clearly semiprime and finite dimensional. We claim that ${}_RM_0$ is also faithful. For each $k > n$, chose M_{i_k} with $1 \leqslant i_k \leqslant n$ and a monomorphism $\mu_k \in \mathrm{Hom}_R(M_{i_k}, M_k)$. Then

$$K = M_0 \oplus M_{i_{n+1}}\mu_{n+1} \oplus \cdots \oplus M_{i_m}\mu_m$$

is an essential submodule of ${}_RM$. By Proposition 1.1 applied to the standard Morita context for ${}_RM$, R is semiprime [prime]. Suppose now that $rM_0 = 0$. Then clearly $rK = 0$ and so $r = 0$ by Lemma 4.2. This establishes the fact that ${}_RM_0$ is faithful. Without loss of generality we can now assume that $M = M_0 = M_1 \oplus \cdots \oplus M_n$.

Since $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$ we can identify $S = \mathrm{Hom}_R(M, M) = \bigoplus_{i=1}^n \mathrm{Hom}_R(M_i, M_i)$ and $N = \mathrm{Hom}_R(M, R) = \bigoplus_{i=1}^n \mathrm{Hom}_R(M_i, R)$. Set $S_i = \mathrm{Hom}_R(M_i, M_i)$ and $N_i = \mathrm{Hom}_R(M_i, R)$. Each N_i is an $S_i - R$-bimodule, and S_i is a left Ore domain (this follows from Theorem 3.5 and the fact that $d({}_{S_i}S_i) = d({}_RM_i) = 1$). Also ${}_{S_i}N_i$ is torsion free. For if $sn = 0$ with $0 \neq s \in S_i$ and $n \in N_i$, then $s[n, M_i] = 0$. Since $[n, M_i] \subseteq S_i$, a left Ore domain, $[n, M_i] = 0$. But then $(M_i, n)^2 = 0$. Since R is semiprime, $(M_i, n) = 0$ and so $n = 0$.

For each $i = 1, \ldots, n$, let Δ_i denote the left quotient division ring of S_i, and set $V_i = \Delta_i \otimes_{S_i} N_i$. V_i is a vector space over Δ_i, and is the injective hull of ${}_{S_i}N_i$. Set $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_n$ and $V = V_1 \oplus \cdots \oplus V_n$; V is the injective hull of ${}_SN$, and we identify $\mathrm{Hom}_\Delta(V, V) = \bigoplus_{i=1}^n \mathrm{Hom}_{\Delta_i}(V_i, V_i)$.

Consider now the Morita context $\mathscr{N} = (S, N, M, R)$ with homomorphisms $[\ ,\ ] : N \otimes_R M \to S$ and $(\ ,\ ) : M \otimes_S N \to R$ where $[\ ,\ ]$ and $(\ ,\ )$ are the homomorphisms of the standard Morita context of ${}_RM$. $\mathscr{N}$ is semiprime. For if $0 \neq n \in N$, then $(M, n) \neq 0$, and since R is semiprime $(M, n)^2 \neq 0$, so that $[n, M]n \neq 0$. Also $Z({}_SN) = 0$ since each ${}_{S_i}N_i$ is torsion free; and N_R is faithful (for if $Nr = 0$ with $r \in R$, then $(rM, N)\,rM = 0$ whence $rM = 0$). Thus we can regard R as a subring of $\mathrm{Hom}_\Delta(V, V)$.

Let ${}_\Delta U \subseteq V$ and $\tau \in \mathrm{Hom}_\Delta(V, V)$ be given with $\dim_\Delta U < \infty$. Then $U_N = U \cap N$ is a finite dimensional submodule of ${}_S N$. By Proposition 3.3 applied to $\mathcal{N}$ there exists $r \in (M, N\tau^{-1} \cap U_N) \subseteq R$ with $l_N(r) \cap U_N = 0$. Then $Nr \subseteq U_N$ and $Nr\tau \subseteq N$. Hence $Vr \subseteq U$ with $\ker r \cap U = 0$. As in the proof of Theorem 3.5 one can show that $r\tau \in R$. With this the proof that R is dense in $\mathrm{Hom}_\Delta(V, V)$ is completed.

(3) *implies* (1). For simplicity we assume that R is a dense subring of $\mathrm{Hom}_\Delta(V, V)$. Let $0 \neq a \in R$ be given and choose ${}_\Delta U \subseteq V$ with $\dim_\Delta U = 1$ and with $a|_U$ monic. Next choose $\tau \in \mathrm{Hom}_\Delta(V, V)$ with $(Ua)\tau = U$. (This is possible since $U \subseteq V_i$ for some $i \leqslant n$, and also $Ua \subseteq V_i$.) Since R is dense in $\mathrm{Hom}_\Delta(V, V)$ there exist $r, s \in R$ with $r\tau = s$ and with $r|_{Ua}$ an automorphism of Ua. Then $Uasa = Uar\tau a = Ua\tau a = Ua$, so that $0 \neq asa \in aRa$. Hence R is semiprime. [The prime case is treated similarly.]

For each $i = 1,\ldots, n$ choose a 1-dimensional subspace U_i of ${}_{\Delta_i}V_i$, and set $U = U_1 \oplus \cdots \oplus U_n$; U is clearly a faithful Δ-module. Set $A = \{r \in R \mid Vr \subseteq U\}$. We will show that A is a faithful finite dimensional left ideal of R.

Given $0 \neq a \in R$, let W be any 1-dimensional subspace of ${}_\Delta Va$, and choose $\tau \in \mathrm{Hom}_\Delta(V, V)$ with $0 \neq W\tau \subseteq U$. Then there exists $r, s \in R$ with $r\tau = s$, $Vr \subseteq W$, and $r|_W$ an automorphism. But then $0 \neq Wr\tau = Ws \subseteq Vas$, and $s \in A$ since $Vs = Vr\tau \subseteq W\tau \subseteq U$. Thus $aA \neq 0$ proving that ${}_R A$ is faithful.

Next suppose that $\sum_{j=1}^m A_j$ is an internal direct sum of nonzero left ideals of R contained in A. For each j, choose $0 \neq a_j \in A_j$ and write $a_j = a_{j1} + \cdots + a_{jn}$ where each $a_{ji} \in \mathrm{Hom}_{\Delta_i}(V_i, V_i)$. If $m > n$, then by the pigeonhole principle at least two distinct a_j have nonzero i-th coordinates for some common i; i.e., $a_{j_1k} \neq 0$, $a_{j_2k} \neq 0$ for some k and $j_1 \neq j_2$. Since each $a_{ji} \in A$ it must be the case that $Va_{j_1k} = U_k = Va_{j_2k}$. Choose $v_1, v_2 \in V$ with $v_1a_{j_1k} = v_2a_{j_2k} \neq 0$. We can assume that $v_1, v_2 \in V_k$, and clearly $V = \Delta v_1 \oplus \ker a_{j_1k}$. Choose $\tau \in \mathrm{Hom}_\Delta(V, V)$ with $v_1\tau = v_2$ and with $(\ker a_{j_1k})\tau = 0$. By the density property there exist $r, s \in R$ with $r\tau = s$, $Vr \subseteq \Delta v_1$, and $r|_{\Delta v_1}$ an automorphism of Δv_1. Now $v_1\tau a_{j_2k} = v_2a_{j_2k} = v_1a_{j_1k}$ and $(\ker a_{j_1k})\tau a_{j_2k} = 0 = (\ker a_{j_1k})a_{j_1k}$, so that $\tau a_{j_2k} = a_{j_1k}$. Hence $0 \neq ra_{j_1k} = r(\tau a_{j_2k}) = sa_{j_2k} \in A_{j_1} \cap A_{j_2}$. Therefore it must be the case that $m \leqslant n$; i.e., $d({}_RA) \leqslant n$. (In fact, it is easy to see that $d({}_RA) = n$. Observe also that this argument shows that A_i is a uniform left ideal of R whenever $A_i \subseteq \mathrm{Hom}_\Delta(V, V_i)$.)

To check that $Z({}_RR) = 0$, suppose not and let $0 \neq a \in Z({}_RR)$. Choose $v \in V$ with $va \neq 0$ and with $\dim_\Delta \Delta v = 1$. $J = \{r \in R \mid Vr \subset \Delta v\}$ is a nonzero left ideal of R, and since $a \in Z({}_RR)$, $J \cap l_R(a) \neq 0$; i.e., there exists $0 \neq r \in J$ with $ra = 0$. Choose $u \in V$ and $\delta \in \Delta$ with $0 \neq ur = \delta v$. Then $0 \neq \delta va = ura$, contradicting $ra = 0$, and so establishing the fact that $Z({}_RR) = 0$. ∎

A prime ring R possesses uniform left ideals and has $Z({}_RR) = 0$ if and only if R has maximal closed left ideals and maximal annihilator left ideals [8, Theorem 1]. Thus the rings of the theorem, in the prime case at least, can be alternatively regarded as rings satisfying maximum conditions.

THEOREM 4.3. *The following conditions on a ring R are equivalent.*

(1) *R is semiprime, $Z({}_RR) = 0$, and R has a faithful left ideal which is an essential extension of a direct sum of uniform left ideals.*

(2) *R has a faithful semiprime module ${}_RM$ with $Z({}_RM) = 0$ and with M an essential extension of a direct sum of uniform submodules.*

(3) *There exist division rings Δ_i, $i \in I$, and Δ_i-vector spaces V_i with R isomorphic to a dense subring of $\prod_{i\in I} \mathrm{Hom}_{\Delta_i}(V_i, V_i)$.*

Proof. The proof of the previous theorem goes through with but minor modifications, which we will now indicate. The finite direct sums are replaced by infinite direct sums or direct products as necessary. The details will be left to the reader.

(2) *implies* (3). Again we can assume that $M = \bigoplus_{i\in I} M_i$ where each ${}_RM_i$ is uniform and $\mathrm{Hom}_R(M_i, M_j) = 0$ for $i \neq j$. Set $S_i = \mathrm{Hom}_R(M_i, M_i)$, $N_i = \mathrm{Hom}_R(M_i, R)$, Δ_i is the left quotient division ring of S_i, $S = \bigoplus_{i\in I} S_i$, $N = \bigoplus_{i\in I} N_i$, $\Delta = \bigoplus_{i\in I} \Delta_i$, $V = \bigoplus_{i\in I} (\Delta_i \otimes {}_{S_i}N_i)$. The only things that must be checked are that (R, M, N, S) is again a semiprime context and that V is the injective hull of ${}_RN$. The verifications are straightforward.

(3) *implies* (1). Here the only difficulty is that the pigeonhole principle argument fails. Instead observe that $A \supseteq \sum_{i\in I} A_i$ where $A_i = \{r \in R \mid Vr \subseteq U_i\}$, and that this sum is direct since each $A_i \subseteq \mathrm{Hom}_{\Delta_i}(V_i, V_i)$. From the original proof we know that each A_i is a uniform left ideal of R. So it suffices to show that A is an essential extension of $\bigoplus_{i\in I} A_i$.

Let $0 \neq a \in A$. Then $0 \neq V_ia = U_i$ for some $i \in I$. Let $W = U_ia^{-1} \cap V_i$; $d({}_\Delta W) = 1$. Use the density property to choose $r \in R$ with $Vr \subseteq W$ and $r|_W$ an automorphism. Then $ra \neq 0$ and $ra \in A_i$ since $Vra \subseteq U_i$, and this proves that ${}_RA$ is an essential extension of $\bigoplus_{i\in I} A_i$. ∎

Remark. In the above theorem, $|I|$ equals the cardinal number of incomparable (i.e., $\mathrm{Hom}_R(A_i, A_j) = 0$) uniform left ideals in a faithful left ideal of R. This number is an invariant for R.

As a special case of the previous theorems we get a generalization of the familiar structure theorem for rings with faithful minimal left ideals.

COROLLARY 4.4. *For a ring R the following conditions are equivalent.*

(1) *R has a faithful completely reducible [minimal] left ideal.*

(2) *There exist division rings Δ_i, $i \in I$, and Δ_i-vector spaces V_i [$|I| = 1$] such that R can be embedded as a classically dense subring of $\prod_{i\in I} \mathrm{Hom}_{\Delta_i}(V_i, V_i)$.*

Proof. (1) *implies* (2). First observe that if R has a faithful completely reducible [minimal] left ideal then R is a semiprime [prime] ring with $Z({}_RR) = 0$. Next note that in view of Corollary 3.4 it suffices to show that the context $\mathcal{N} = (S, N, M, R)$ in the proof that (2) *implies* (3) of the previous theorem satisfies: given $n \in N$ there exists $m \in M$ with $[n, m]n = n$.

Using the fact that R is semiprime and that ${}_RM$ is a completely reducible [minimal] left ideal we can assume $M = \bigoplus_{i\in I} Re_i$ $[|I| = 1]$ where $e_i = e_i^2 \in R$ and $e_iRe_j = 0$ for $i \neq j$. Also we can identify $N = \bigoplus_{i\in I} e_iR$, $S = \bigoplus_{i\in I} e_iRe_i$, where each e_iRe_i is a division ring and the maps belonging to the Morita context $\mathcal{N}$ are just multiplications in R. Now let $0 \neq n \in N$ be given, and for simplicity write $n = e_1a_1 + \cdots + e_ta_t$ with each $e_ia_i \neq 0$. Since each e_iRe_i is a division ring we can choose elements $b_1, \ldots, b_t \in R$ with $(e_ia_ie_i)(e_ib_ie_i) = e_i$ for $i = 1, \ldots, t$. Set $m = \sum_{i=1}^t e_ib_ie_i \in M$. Then $[n, m]n = [\sum_{i=1}^t e_ia_i, \sum_{i=1}^t e_ib_ie_i]n = (\sum_{i=1}^t e_ia_ie_ib_ie_i)n = (\sum_{i=1}^t e_i)n = n$.

(2) *implies* (1). We assume that R is a classically dense subring of $\prod_{i\in I} \mathrm{Hom}_{\Delta_i}(V_i, V_i)$ $[|I| = 1]$. Then (notation as in the proof of Theorem 4.3) we know that each $A_i = \{r \in R \mid Vr \subseteq U_i\}$ is a uniform left ideal of R and $A = \{r \in R \mid Vr \subseteq U\}$ is an essential extension of $\bigoplus_{i\in I} A_i$ and a faithful left ideal of R. Also R is a semiprime ring.

For any fixed $i \in I$ use the density property to choose $e_i \in R$ with $e_i|_{U_i} = 1_{U_i}$ and $Ve_i \subseteq U_i$. We claim that e_iR is a minimal right ideal of R. For any $a \in R$ with $e_ia \neq 0$, it is the case that $\dim_\Delta Ve_ia = 1$ and $Ve_ia \subseteq V_i$. Let $0 \neq u \in U_i$ and choose any $\tau \in \mathrm{Hom}_\Delta(V, V)$ with $(ue_ia)\tau = u$. By the density property there exist $b, c \in R$ with $c\tau = b$ and $c|_{\Delta ue_ia} = 1_{\Delta ue_ia}$. Hence $ue_iab = ue_iac\tau = ue_ia\tau = u = ue_i$. Also $\ker e_iab \supseteq \ker e_i$ with the latter a maximal submodule of ${}_\Delta V$. Hence $\ker e_iab = \ker e_i$, and it follows that $e_iab = e_i$ and so $e_i \in e_iaR$. This proves that e_iR is a minimal right ideal of R.

Since $e_i^2 = e_i \in A_i$ and R is semiprime, Re_i is a minimal left ideal of R [9, p. 63] contained in the uniform left ideal A_i. From this it follows that $B = \bigoplus_{i\in I} Re_i$ is an essential submodule of ${}_RA$. By Lemma 4.2, ${}_RB$ is a faithful completely reducible [minimal] left ideal of R. ∎

Observe that classically dense subrings in the above sense have the property that given any finite dimensional subspace U of ${}_\Delta V$ there exists $e = e^2 \in R$ with $Ve = U$. This is a priori stronger than the "classical" density statement [6, p. 75].

The prime case of Theorem 4.1 appears in [1], and it is from that source that the inspiration for this paper derives. Our approach is however somewhat different in spirit and substance, with Proposition 3.3 the basic step.

Contrary to the suggestion made in [1, Remark 10A], the semiprimeness of R cannot be dropped as a hypothesis in Theorem 4.1 (1). For an example, let R be the ring of all $n \times n$ lower triangular matrices over a field. We do not however have an example to show that the hypothesis $Z({}_RR) = 0$ is necessary.

When R is a dense subring of $\mathrm{Hom}_\Delta(V, V)$, the ideal $R_i = \{r \in R \mid Vr \subseteq V_i\}$ can be regarded as a dense subring of $\mathrm{Hom}_{\Delta_i}(V_i, V_i)$ and $\bigoplus_{i \in I} R_i$ is an essential left ideal of R. Thus much information can be readily extended from the case of prime dense rings. See [1], especially Theorem 10C. We will not treat these routine extensions here, with the exception of the following result, which illuminates the connection with semiprime noetherian rings (and requires but a brief proof).

By T_n for a ring T one denotes the ring of $n \times n$ matrices over T. A subring S of a ring T is called a *left order* in T if T is a classical ring of left quotients for S.

THEOREM 4.5. *Let R, V be as in Theorem 4.2, and for each $i \in I$ set $R_i = \{r \in R \mid Vr \subseteq V_i\}$. Then either $\dim_{\Delta_i} V_i = n_i < \infty$, in which case R_i is isomorphic to a left order in $(\Delta_i)_{n_i}$; or $\dim_{\Delta_i} V_i = \infty$, in which case R_i contains left ideals R_{ik} of R, one for each positive integer k, and there exists a ring homomorphism of R_{ik} onto a left order in $(\Delta_i)_k$.*

Proof. For any integer $1 \leqslant k \leqslant \dim_{\Delta_i} V_i$, choose a k-dimensional subspace W of ${}_\Delta V_i$, and set $R_{ik} = \{r \in R \mid Vr \subseteq W\}$. Define $\varphi_{ik} : R_{ik} \to \mathrm{Hom}_{\Delta_i}(W, W)$ to be the canonical map of R_{ik} onto $\langle R_{ik}\rangle = R_{ik}/\{r \in R_{ik} \mid Wr = 0\}$ restricted to W. We can regard $\langle R_{ik}\rangle$ as a subring of $\mathrm{Hom}_{\Delta_i}(W, W) \cong (\Delta_i)_k$. Given $\tau \in \mathrm{Hom}_{\Delta_i}(W, W)$, extend τ to an element τ' of $\mathrm{Hom}_\Delta(V, V)$. The density property ensures us that there exists $r \in R_{ik}$ with $r\mid_W$ an automorphism of ${}_\Delta W$ and with $r\tau' = s \in R_{ik}$. Then $r\mid_W \tau'\mid_W = s\mid_W$ and so $\tau = \langle r\rangle^{-1}\langle s\rangle$. Also by the density property, regular elements of $\langle R_{ik}\rangle$ are represented by automorphisms of W. Therefore $\langle R_{ik}\rangle$ is a left order in $\mathrm{Hom}_{\Delta_i}(W, W)$. ∎

We remark that in the classical setting (i.e., Corollary 4.4) $\langle R_{ik}\rangle = \mathrm{Hom}_\Delta(W, W)$. Also if $d({}_RR) = d({}_SN) = d({}_\Delta V) = n < \infty$ then each R_i is a left order in $(\Delta_i)_{n_i}$ with $\sum n_i = n$.

5. We next treat dense rings of transformations from the viewpoint of dual vector spaces [6, p. 75]. The case of a prime dense ring is treated in Theorem 4, Lemma 7, and Corollary 8 of [1].

PROPOSITION 5.1. *Suppose that (R, M, N, S) is a semiprime Morita context and that K is a finite dimensional submodule of ${}_RM$ with $Z({}_RK) = 0$. Then there exists $r \in R^1$, $s' = \sum_{j=1}^{t} [n_j, m_j'] \in S$, $m_j, m_j' \in K$, $n_j \in N$, such that*

(1) $\sum_{j=1}^{t} Rm_j$ and $\sum_{j=1}^{t} Rm_j'$ *are direct sums of uniform modules and are essential submodules of ${}_RK$;*

(2) $(m_j, n_k)\, m_k' = \delta_{jk} r m_j'$, *where δ_{jk} is the Kronecker delta;*

(3) $rm_j' = m_j s' \neq 0$ *for* $j = 1,\ldots, t$;

(4) $l_M(s') \cap K = 0$ *and* Ks' *is an essential submodule of* ${}_RK$. *When* ${}_RK$ *is injective we can choose* $r = 1$.

Proof. By Proposition 3.3 and the remark following it, there exists an element $s' = \sum_{k=1}^t [n_k, m_k'] \in S$ where $n_k \in N$, $m_k' \in K$, such that $\sum_{j=1}^t Rm_j'$ is a direct sum and an essential submodule of ${}_RK$, Ks' is an essential submodule of ${}_RK$, and $l_M(s') \cap K = 0$. Since ${}_R\hat{K}$ is finite dimensional and injective, $\hat{K}s' = \hat{K}$ (really we mean the unique extension of $s' |_K$ to $\mathrm{Hom}_R(\hat{K}, \hat{K})$, but no confusion should arise). Hence there exist elements $\hat{m}_j \in \hat{K}$ with $\hat{m}_j s' = m_j'$. Choose $r \in R^1$ with $0 \neq r\hat{m}_j = m_j \in K$ for each $j = 1,\ldots, t$. [Note that if $K = \hat{K}$, we take $r = 1$.] Thus $0 \neq m_j s' = r\hat{m}_j s' = rm_j'$ for each j.

We claim that $\sum_{j=1}^t Rm_j$ is direct. For if $\sum_{j=1}^t r_j m_j = 0$ then $0 = \sum_{j=1}^t r_j m_j s' = \sum_{j=1}^t r_j r m_j'$. Since $\sum_{j=1}^t Rm_j'$ is direct, $0 = r_j r m_j' = r_j m_j s'$ for each i. Since $l_M(s') \cap K = 0$, each $r_j m_j = 0$.

Next consider $rm_j' = m_j s' = \sum_{k=1}^t m_j[n_k, m_k'] = \sum_{k=1}^t (m_j, n_k) m_k'$. Since $\sum_{j=1}^t Rm_j'$ is direct, we have $rm_j' = (m_j, n_j) m_j'$ and $(m_j, n_k) m_k' = 0$ when $j \neq k$. The rest is easily verified. ▮

PROPOSITION 5.2 (Dual Basis Lemma). *Let* $R = \bigoplus_{i\in I} R_i$ *where each* R_i *is a left Öre domain, and for each* $i \in I$ *suppose that* ${}_{R_i}M_i$ *is a torsion-free module. Set* $M = \bigoplus_{i\in I} M_i$. *Let* $\mathscr{M} = (R, M, N, S)$ *be a Morita context satisfying* $(m, N) \neq 0$ *for* $m \neq 0$. *Then given* ${}_RK \subseteq {}_RM$ *with* $d({}_RK) < \infty$ *there exist* $r \in R^1$, $s = \sum_{j=1}^t [n_j, m_j]$, $n_j \in N$, $m_j \in K$ *such that*

(1) $\sum_{j=1}^t Rm_j$ *is a direct sum of uniform modules and is an essential submodule of* ${}_RK$;

(2) $(m_j, n_k) m_k = \delta_{jk} rm_j$;

(3) $rm_j = m_j s \neq 0$ *for* $j = 1,\ldots, t$;

(4) $l_M(s) \cap K = 0$ *and* Ks *is an essential submodule of* ${}_RK$. *When* ${}_RK$ *is injective we can choose* $r = 1$.

Proof. Note that $\mathscr{M}$ is a semiprime context. For if $0 \neq m \in M$, then $0 \neq (m, N) \subseteq R$ and so $0 \neq (m, N)^2 \subseteq ((m, N)m, N)$; in particular $(m,N)m \neq 0$. Also $Z({}_RM) = 0$ since each ${}_{R_i}M_i$ is torsion-free. Hence we can choose r, s', m_j, m_j', n_j satisfying the previous proposition.

For each $j = 1,\ldots, t$, there exists $i_j \in I$ with $m_j' \in M_{i_j}$. (This is because ${}_RRm_j'$ is uniform.) Since $rm_j' = m_j s'$ and Rm_j is uniform, it must be the case that $m_j \in M_{i_j}$. Thus $r_{i_j} m_j' = rm_j' = (m_j, n_j) m_j' = (m_j, n_j)_{i_j} m_j'$ where a_{i_j} denotes the i_j-coordinate (in R_{i_j}) of $a \in R$. Since each ${}_{R_i}M_i$ is torsion-free $r_{i_j} = (m_j, n_j)_{i_j}$, and then $rm_j = r_{i_j} m_j = (m_j, n_j)_{i_j} m_j = (m_j, n_j) m_j$. Similarly one shows that $(m_j, n_k) m_k = 0$ for $j \neq k$. Set $s = \sum_{k=1}^t [n_k, m_k]$. Then $m_j s = \sum_{k=1}^t (m_j, n_k) m_k = (m_j, n_j) m_j = rm_j$. The rest is a routine check. ▮

Remark. When $|I| = 1$, statement (2) can be simplified to $(m_j, n_k) = \delta_{jk}r$. When R is a division ring (or a direct sum of division rings) r can be chosen equal to 1 and then $(m_j, n_k) = \delta_{jk}$. So this reduces to the classical dual basis statement.

COROLLARY 5.3. *If R is a dense subring of* $\mathrm{Hom}_\Delta(V, V)$ *as in Section* 4, *then given* $\tau \in \mathrm{Hom}_\Delta(V, V)$ *and a finite dimensional submodule U of ${}_\Delta V$ there exist $a, b \in R$ with $a\tau = b$, $Va \subseteq U$, $a|_U$ an automorphism of U; moreover, a can be chosen so that $u_i a = \delta u_i$ for some Δ-basis $u_1, \ldots, u_t$ of U and $\delta \in \Delta$.*

Proof. Apply the preceding proposition to the context $\mathcal{N}$ in the last paragraph of the proof that (2) *implies* (3) of Theorem 4.1. ▮

When a context $\mathcal{M} = (R, M, N, S)$ satisfies the hypothesis of the Dual Basis Lemma with M_S faithful, we call $\mathcal{M}$ a *dual context* for S. When $\mathcal{M}$ is a dual context for S, S can therefore be regarded as a subring of $\mathrm{Hom}_R(M, M)$. By a *full linear ring* we mean the ring of linear transformations of a (left) vector space over a division ring. In this terminology, the proof of the previous corollary actually shows the following.

COROLLARY 5.4. *R is a dense subring of a direct product of full linear rings if and only if there exists a dual context (S, N, M, R) for R. Moreover, writing $S = \bigoplus_{i \in I} S_i$, $N = \bigoplus_{i \in I} N_i$, where each ${}_{S_i}N_i$ is torsion free over the left Öre domain S_i, R is (isomorphic to) a dense subring of* $\prod_{i \in I} \mathrm{Hom}_{\Delta_i}(V_i, V_i)$ *(where Δ_i is the left quotient division ring of S_i and $V_i = \Delta_i \otimes_{S_i} N_i$ for each $i \in I$) in the following sense. Given any* $\tau \in \mathrm{Hom}_\Delta(V, V)$ *and ${}_\Delta U \subseteq V$ with* $\dim {}_\Delta U < \infty$, *there exists a basis $\{u_j \mid j = 1, \ldots, t\}$ for ${}_\Delta U$ contained in N, and there exists $a, b \in R$, $s \in S$ such that $a\tau = b$ and $u_j a = s u_j \neq 0$ for each $j = 1, \ldots, t$.*

For a discussion of the uniqueness of a dual context for a (prime) dense ring of linear transformations, see Theorem 11 of [1].

The next result extends Theorem 10C (5) of [1]. It is also possible to obtain it from a knowledge of the prime case, but we include a proof for the sake of completeness.

THEOREM 5.5. *Let R be a dense subring of a direct product of full linear rings. Then $d({}_R R) \leqslant d(R_R)$, with equality holding when $d(R_R) < \infty$.*

Proof. Choose a dual context $\mathcal{N} = (S, N, M, R)$ for R as in the proof of (2) *implies* (3) of Theorem 4.3; i.e., $M = \bigoplus_{i \in I} M_i$, $S = \bigoplus_{i \in I} S_i$ where each $S_i = \mathrm{Hom}_R(M_i, M_i)$, and $N = \bigoplus_{i \in I} N_i$ where each $N_i = \mathrm{Hom}_R(M_i, R)$.

Since $\mathscr{N}$ is a dual context, $\mathscr{N}$ and $\mathscr{N}^\circ = (S^\circ, M^\circ, N^\circ, R^\circ)$ are both semiprime contexts and both N_R and $M^\circ_{R^\circ}$ are faithful. By Proposition 2.4 (2) applied to $\mathscr{N}$ and $\mathscr{N}^\circ$, $d({}_RR) = d({}_SN)$ and $d(R_R) = d({}_{R^\circ}R^\circ) = d({}_{S^\circ}M^\circ) = d(M_S)$. So it suffices to show that $d({}_SN) \leqslant d(M_S)$.

Suppose that K is a submodule of ${}_SN$ with $d({}_SK) = t < \infty$. Use the dual basis lemma to choose $n_1, ..., n_t \in K$ and $m_1, ..., m_t \in M$ with each Sn_j a uniform submodule of ${}_SN$ and with $[n_j, m_k]\, n_k = \delta_{jk} sn_j$ for some $s \in S$, with $sn_j \neq 0$, $j = 1, ..., t$. For each $j = 1, ..., t$ choose $i_j \in I$ with $n_j \in N_{i_j}$ (as in the dual basis lemma). Since (,) is the evaluation map of the standard Morita context for ${}_RM$ and $N_{i_j} = \mathrm{Hom}_R(M_{i_j}, R)$, $\delta_{jk} sn_j = [n_j, m_k]\, n_k = n_j(m_k, n_k) = n_j((m_k)_{i_k}, n_k)$ where $(m_k)_{i_j}$ is the "M_{i_j}-coordinate of m_k" as an element of $M = \bigoplus_{i \in I} M_i$. Hence we can assume without loss of generality that each $m_j = (m_j)_{i_j} \in M_{i_j}$. It follows that $0 \neq [n_j, m_j]_{i_j} = s_{i_j} \in S_{i_j}$ and $[n_j, m_k]_{i_k} = 0 \in S_{i_k}$ for $j \neq k$.

The proof is completed by showing that $\sum_{j=1}^t m_j S = \sum_{j=1}^t m_j S_{i_j}$ is a direct sum of S-submodules of M. Suppose that $\sum_{j=1}^t m_j s_j = 0$ with each $s_j \in S_{i_j}$. Then $0 = [n_k, \sum_{j=1}^t m_j s_j] = \sum_{j=1}^t [n_k, m_j]\, s_j$, so $\sum_{j=1}^t [n_k, m_j]_{i_j} s_j = 0 \in S_{i_j}$. That is, $[n_k, m_k]_{i_k} s_k = s_{i_k} s_k$. Since each S_{i_k} is an Öre domain and $s_{i_k} \neq 0$, it is the case that $s_k = 0$ for $k = 1, ..., t$.

When $d(R_R) < \infty$, R has semisimple artinian classical left and right quotient rings, which are well known to coincide. Thus $d({}_RR) = d(R_R)$ in this case. ∎

6. It is also possible to determine when certain endomorphism rings are dense rings of linear transformations. The following result is a generalization of Theorem 3.5.

THEOREM 6.1. *Let $\mathscr{M} = (R, M, N, S)$ be a Morita context satisfying $(M, N)m \neq 0$ for all $0 \neq m \in M$. Then $S/\mathrm{ann}\, M_S$ is isomorphic to a dense subring of a [direct product of] full linear ring[s] if and only if $\mathscr{M}$ is prime [semiprime], $Z({}_RM) = 0$, and ${}_{\langle R\rangle}M$ contains a faithful direct sum of uniform submodules, where $\langle R\rangle = R/\mathrm{ann}_R M$.*

Proof. In view of Theorem 4.1 (and Propositions 1.1, 2.4, and Lemma 4.2), it suffices to show that $\langle S\rangle = S/\mathrm{ann}\, M_S$ contains a faithful direct sum of uniform left ideals when and only when ${}_{\langle R\rangle}M$ contains a faithful direct sum of uniform submodules.

Suppose that $\langle A\rangle$ is a faithful left ideal of $\langle S\rangle$, where $\langle A\rangle = \sum_{\gamma \in \Gamma} \langle A_\gamma\rangle$ is a direct sum of uniform left ideals $\langle A_\gamma\rangle$. By Proposition 2.4 (2) and its proof, $M_0 = \sum_{\gamma \in \Gamma} MA_\gamma$ is a direct sum of uniform submodules MA_γ of ${}_RM$, where $A_\gamma = \{s \in S \mid \langle s\rangle \in \langle A_\gamma\rangle\}$. It remains to check that ${}_{\langle R\rangle}M_0$ is faithful. If $\langle 0\rangle \neq \langle r\rangle \in \langle R\rangle$, then $rM \neq 0$. Since $\mathscr{M}$ is semiprime, $rM[N, rM] \neq 0$, and in particular $\langle[N, rM]\rangle \neq \langle 0\rangle$. Since ${}_{\langle S\rangle}\langle A\rangle$ is faithful, $\langle 0\rangle \neq \langle[N, rM]\rangle\langle A\rangle$, and so $0 \neq rM\langle A\rangle \subseteq rM_0$.

Conversely, suppose that $M_0 = \sum_{\gamma\in\Gamma} M_\gamma$ is a direct sum of uniform submodules M_γ of ${}_RM$, ${}_{\langle R\rangle}M_0$ faithful. Then by Proposition 2.4 (1) and its proof, $\langle A\rangle = \sum_{\gamma\in\Gamma}\langle[N, M_\gamma]\rangle$ is a direct sum, and each $\langle[N, M_\gamma]\rangle$ is a uniform left ideal of $\langle S\rangle$. To see that $\langle A\rangle$ is a faithful left ideal of $\langle S\rangle$, assume that $\langle s\rangle\langle A\rangle = \langle 0\rangle$ for some $s \in S$. Then for each $\gamma \in \Gamma$, $\langle s\rangle\langle[N, M_\gamma]\rangle = \langle 0\rangle$; i.e., $0 = Ms[N, M_\gamma] = (Ms, N)\, M_\gamma$. Hence $(Ms, N)\, M_0 = 0$. Since ${}_{\langle R\rangle}M_0$ was faithful, $(Ms, N)M = 0$, and so $Ms = 0$. That is $\langle s\rangle = \langle 0\rangle$. ∎

THEOREM 6.2. *Suppose that R is a dense subring of a (direct product of; finite direct sum of) full linear ring(s), and that ${}_RM$ is a torsionless left R-module. Then $E(M) = \mathrm{Hom}_R(M, M)$ is a dense subring of a (direct product of; finite direct sum of) full linear ring(s).*

Proof. (Sketch) Let ${}_RM$ be a torsionless module where R is dense in $\mathrm{Hom}_\Delta(V, V) = \prod_{i\in I}\mathrm{Hom}_{\Delta_i}(V_i, V_i)$. Then $Z({}_RM) = 0$ and ${}_RM$ is semiprime. Set $R_i = \{r \in R \mid Vr \subseteq V_i\}$. As we noted earlier, each R_i is an ideal of R and a prime dense subring of $\mathrm{Hom}_{\Delta_i}(V_i, V_i)$, and $\bigoplus_{i\in I} R_i$ is an essential left ideal of R.

Set $J = \{j \in I \mid R_jM \neq 0\}$, $R_0 = \bigoplus_{i\in I\setminus J} R_i$, $R_1 = \bigoplus_{j\in J} R_j$. For each $j \in J$, R_jM contains a submodule isomorphic to a uniform left ideal of R_j. To see this, choose a uniform left ideal U_j of R_j and $m \in M$ with $U_jm \neq 0$. Then $U_jm \simeq U_j$.

Set $M_1 = \sum_{j\in J} R_jM$. This sum is direct since $R_jR_k = 0$ whenever $j \neq k \in I$; and M_1 is a faithful essential R_1-submodule of M. Moreover, given any $f \in \mathrm{Hom}_R(M, R)$, $f|_{M_1} : M_1 \to R_1$ because $M_1f = (\sum_{j\in J} R_jM)f = \sum_{j\in J} R_j(Mf) \subseteq \sum_{j\in J} R_j = R_1$. It follows that M_1 is a semiprime R_1-module. Finally, using the fact that $R_0 \oplus R_1$ is an essential left ideal of R, one checks that $Z({}_{R_1}M_1) = 0$. The conclusion now follows by applying the previous theorem to the standard Morita context for ${}_{R_1}M_1$. ∎

As an immediate consequence, one has an extension of Theorem 2.3 of [12].

COROLLARY 6.3. *Let ${}_RM$ be a torsionless module over a ring R which has a simple (semisimple) artinian classical left quotient ring. Then $E(M) = \mathrm{Hom}_R(M, M)$ is a dense subring of a full linear ring (of a finite direct sum of full linear rings).*

It is perhaps worth pointing out that a very special case of this corollary yields the (possibly unknown) fact that the endomorphism ring of a torsionless abelian group is a dense ring of linear transformations (of a vector space over the rational numbers).

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